Problem Set 1
Assigned: Tuesday, Sep. 31, 2004
Due: Tuesday, Sep. 14, 2004

Problem 1.1: A fair coin is flipped eight times.

a) Describe an appropriate probability space \((\Omega, \mathcal{F}, P)\) corresponding to observing the outcomes (i.e. Heads, Tails) of the coin flips.

b) Express each of the following events explicitly as subsets of \(\Omega\) and find their probabilities.

\(E_1 = \) The first four flips are Heads and the last four flips are Tails.
\(E_2 = \) The first four flips are Heads.
\(E_3 = \) Exactly four of the flips are Heads.
\(E_4 = \) At least four Heads in a row occur.

Problem 1.2: Let \(X\) be a random number uniformly distributed in the interval \([0, 1]\). Find increasing functions \(g\) and \(h\) so that the random variable \(Y = g(X)\) is exponentially distributed with mean one and the random variable \(Z = h(X)\) has the Cauchy density

\[ f_Z(z) = \frac{1}{\pi(1+z^2)}. \]

Problem 1.3: Stark & Woods 1.30

Problem 1.4: Let \(X\) be exponentially distributed with mean \(\lambda^{-1}\). Find the distribution functions (and carefully sketch them) for the random variables \(Y = e^X\) and \(Z = \min(X, 3)\).

Problem 1.5: Let \(X\) have probability density function \(f_X\) where \(f_X(x) = e^{-x}\) for \(x \geq 0\) and \(f_X(x) = 0\) for \(x < 0\), and let \(Y = 1 + \sqrt{X}\).

a) Find the cumulative probability distribution function of \(Y\) and sketch it.

b) Find the probability density function of \(Y\) and sketch it.

Problem 1.6: Suppose the length \(L\) and width \(W\) of a rectangle are each uniformly distributed over the interval \([0, 1]\). Let \(C = 2L + 2W\) (the length of the perimeter) and \(A = LW\) (the area).

a) Find the means, variances, and probability densities of \(C\) and \(A\).

b) Find the correlation coefficient between \(A\) and \(L\) (recall that \(A = LW\)).

c) Find the conditional density of \(A\) given \(L\). Be careful to specify the function everywhere it is defined. (Hint: there is a hard way and an easy way to solve this one.)

Problem 1.7: Let \(X_1\) and \(X_2\) be zero-mean jointly Gaussian random variables with covariance matrix

\[ A_X = \begin{bmatrix} 14/5 & -2/5 \\ -2/5 & 11/5 \end{bmatrix} \] (1)

a) Find the marginal probability density function of \(X_1\).

b) Find the probability density function of \(Y = 2X_1 + X_2\).
Problem 1.8: Let $X_1, X_2, ..., X_n, ...$ be a sequence of random variables with identical means $E[X_i] = m_X$, and variances $Var(X_i) = \sigma^2$. We define the sampled mean and the sampled mean-square of the first $N$ of the $X_i$’s as

$$m(N) = \frac{1}{N} \sum_{i=1}^{N} X_i = \text{sampled mean}$$

$$s(N) = \frac{1}{N} \sum_{i=1}^{N} X_i^2 = \text{sampled mean-square}.$$  \hspace{1cm} (2)

a) Suppose that the $X_i$’s are uncorrelated random variables. Find the mean and variance of the sampled mean. Show that $\lim_{N \to \infty} E[(m(N) - m_X)^2] = 0$.

b) Suppose that the $X_i$’s are statistically independent zero-mean Gaussian random variables. Find the mean and variance of the sampled mean-square. Show that $\lim_{N \to \infty} E[(s(N) - \sigma^2)^2] = 0$.

Problem 1.9: Let $X$ and $Y$ be two jointly Gaussian random variables which have zero-mean, i.e. $E[X] = E[Y] = 0$. Let $Z_1 = aY + b$, $Z_2 = Y^2$, $Z_3 = Y^3$, $Z_4 = \cos(Y)$. Compute $E[X|Z_i]$ for $i = 1, 2, 3, 4$ and relate the expression you obtain to $E[X|Y] = \frac{\sigma_{XY}}{\sigma_{YY}}Y$.

Problem 1.10: If $X$ is a real-valued random variable on $(\Omega, \mathcal{F}, P)$ and $g$ is a Borel measurable function, then prove that $Y$ defined by $Y = g(X)$ is also a random variable on $(\Omega, \mathcal{F})$.

Problem 1.11: Let $X$ and $Y$ be real random variables. Prove that their sum $Z = X + Y$ is also a random variable. (Hint: Use the definition of random variables are mapping on $\sigma$-algebra and the countable union and intersection properties of measurable sets.)